A Quantum Multiparty Packing Lemma and the Relay Channel

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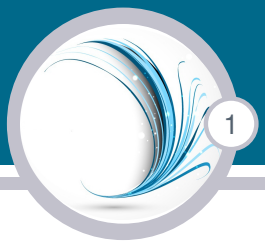
Joint with Hrant Gharibyan, Patrick Hayden, and Michael Walter

Outline



Introduction
Message packing

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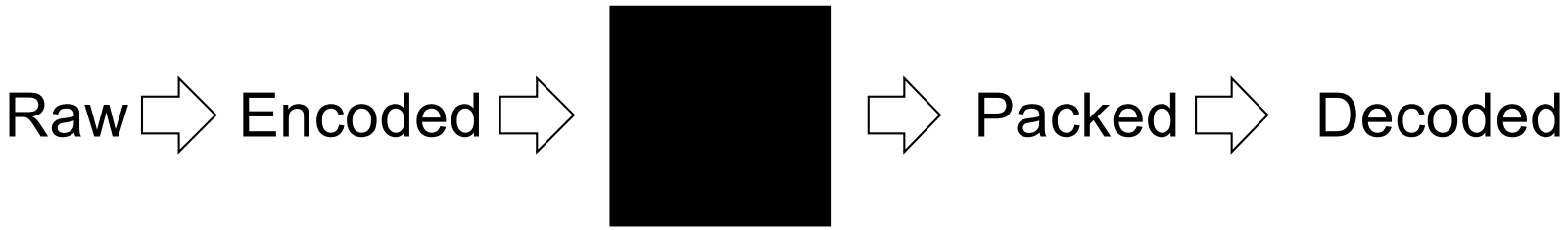
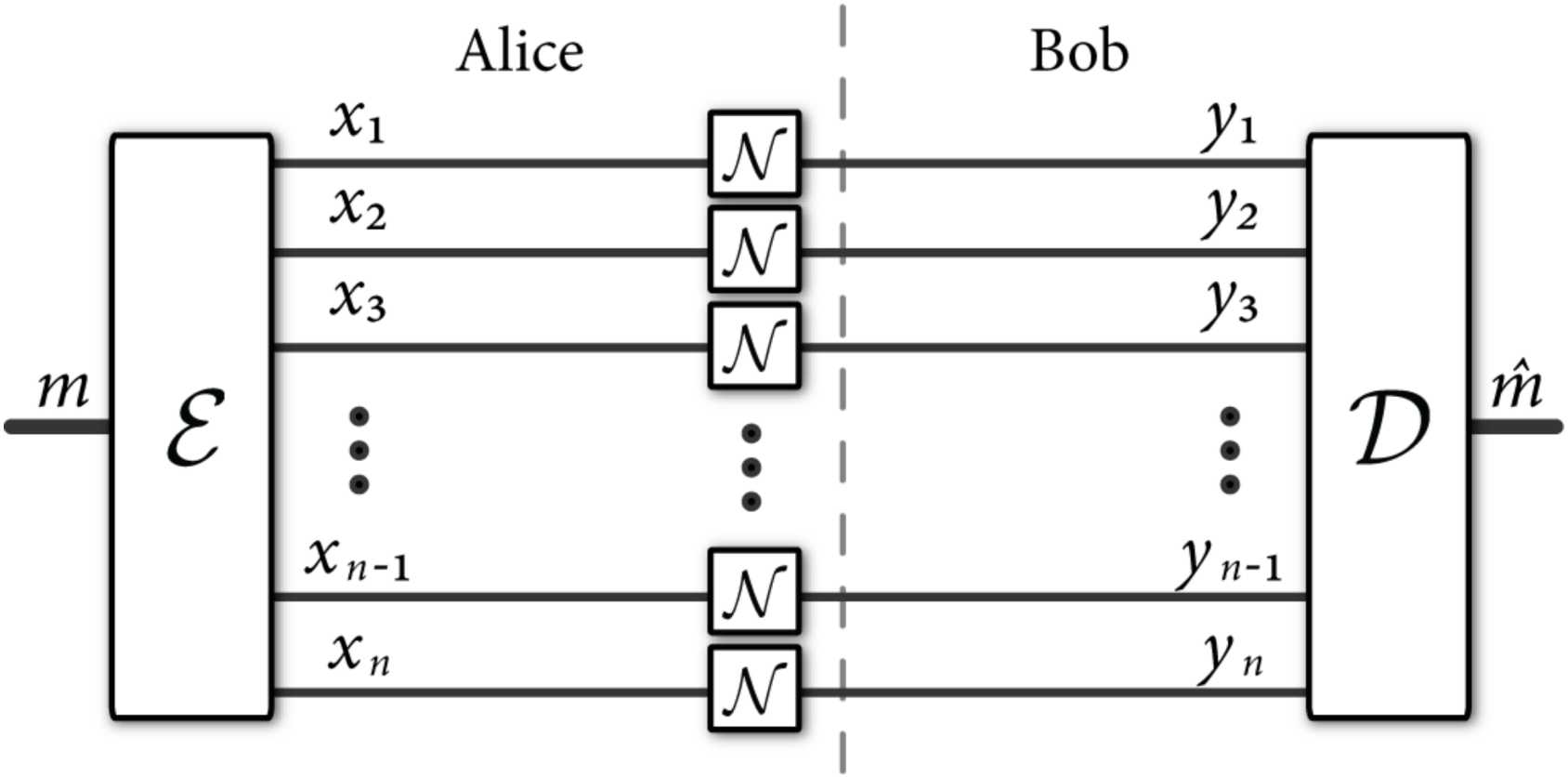


Introduction
Communication as message packing



Taken from Mark Wilde's From Classical to Quantum Shannon Theory

Introduction
Encoding messages into quantum systems

- Classical-quantum black box: Input is classical, output is quantum

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Theorem (Holevo-Schumacher-Westmoreland theorem)

The classical capacity of a quantum channel $\mathcal{N}_{A\to B}$ with separable encodings is

$$C(\mathcal{N}) = \max_{\{p_X, \rho_A^{(x)}\}} I(X;B)_\rho$$

where

$$\rho_{XB} \equiv \sum_x p_X(x) \left|x\right\rangle \left\langle x\right|_X \otimes \rho_B^{(x)}.$$

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- Network information theory: multiple senders and/or multiple receivers
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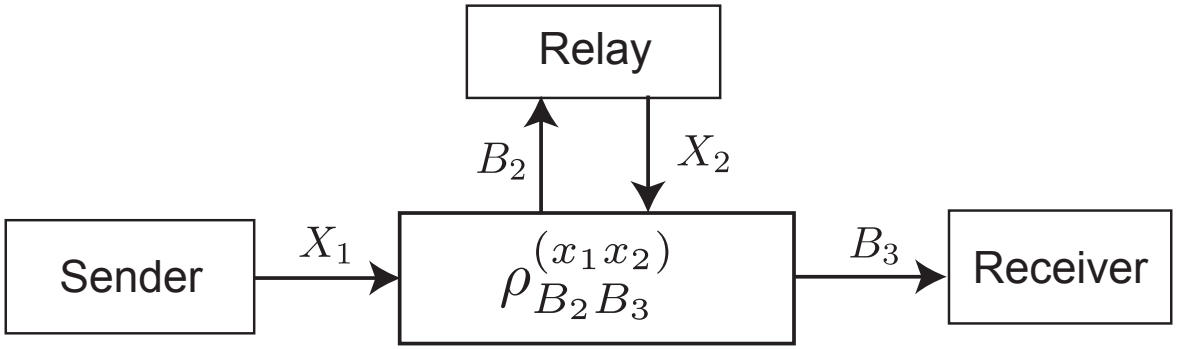
  - Mostly open for quantum channels, especially one-shot
- Multiparty packing: encoding multiple messages M_j into a quantum system B via multiple classical systems X_v
  - Multiple senders: multiple access channel, relay channel
- Assuming classical-quantum channel, codebook is of the form $\{x_v(m)\}_{v\in V, m\in M}$, where $M = \bigtimes_{j\in J} M_j$

Relay channel
Relay channel definition



- $\mathcal{N}_{X_1X_2\to B_2B_3} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_3}, (x_1, x_2) \mapsto \rho_{B_2B_3}^{(x_1x_2)}$ (SWV, 2012)

Relay channel
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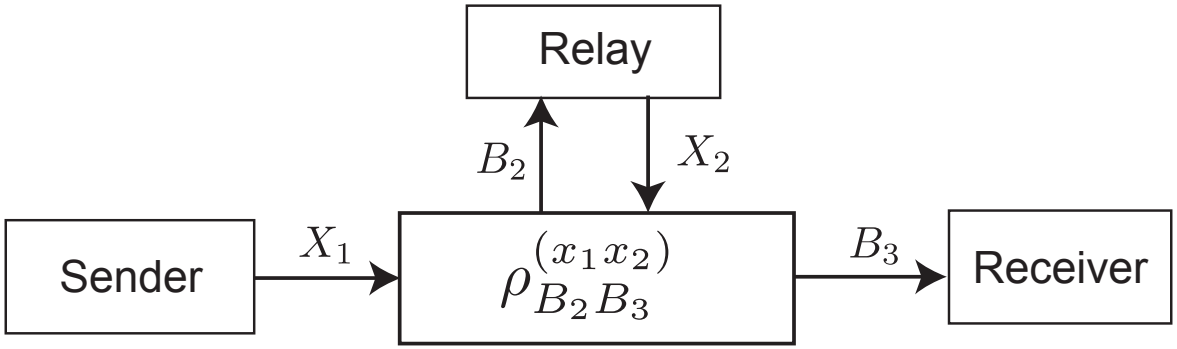

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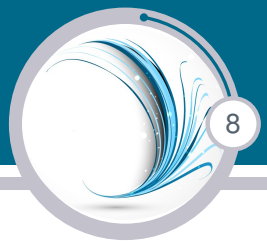
- Relay's transmission affects relay's system, sender's transmission affects receiver's: More general than concatenated channels!

Relay channel definition
Multihop bound

Quantum generalization of classical protocols

Relay channel definition
Multihop bound



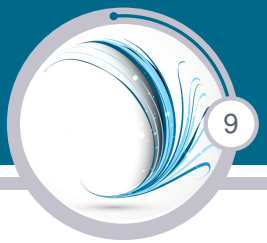
Quantum generalization of classical protocols

- Multihop

Relay channel definition
Multihop bound

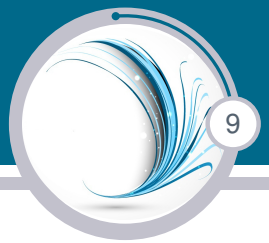
Quantum generalization of classical protocols

- Multihop
  - Treat relay channel as concatenated channel: sender transmits message to relay, relay transmits decoded message to receiver



Random codebook

$$C = \bigcup_{j=1}^{b} \{(x_1)_j(m_j), (x_2)_j(m_{j-1})\}_{m_j \in M_j, m_{j-1} \in M_{j-1}}$$



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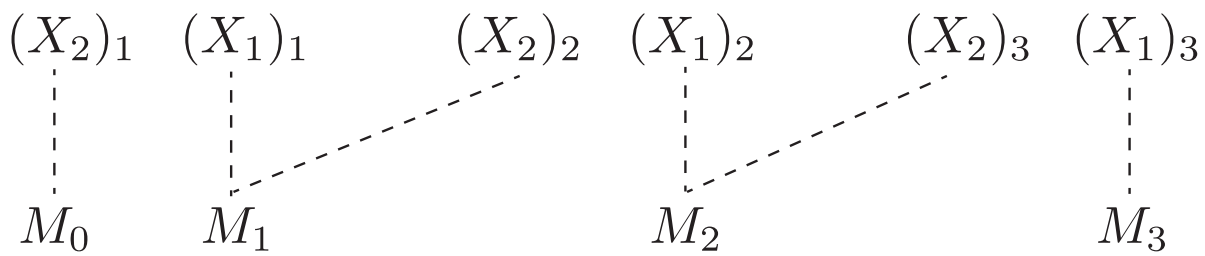
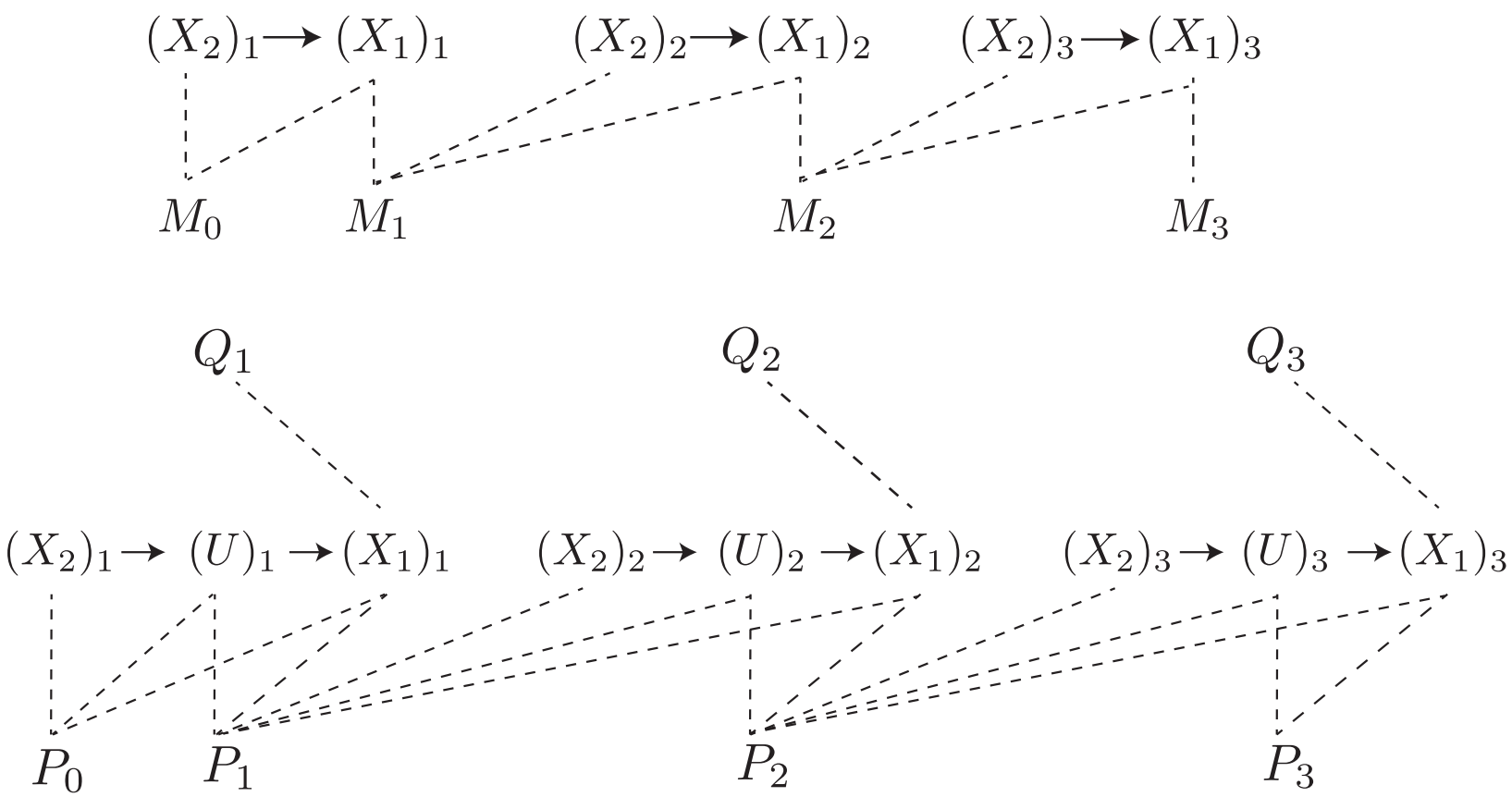
Code:



Relay channel
Remarks

- Coherent multihop, decode forward, partial decode forward



- All straightforward generalizations of classical protocols

Random codebooks for network communication

Quantum multiparty packing lemma
Multiplex Bayesian networks

Random codebooks for network communication

- Has structure depending on network setting and protocol chosen

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  - Muliplex Bayesian network

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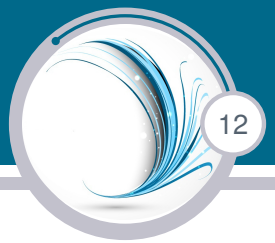

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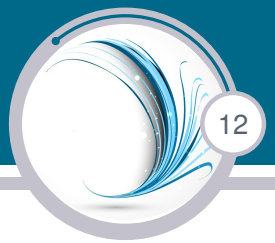

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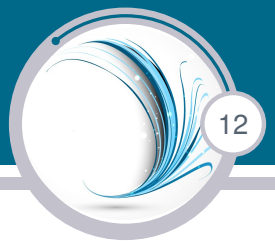

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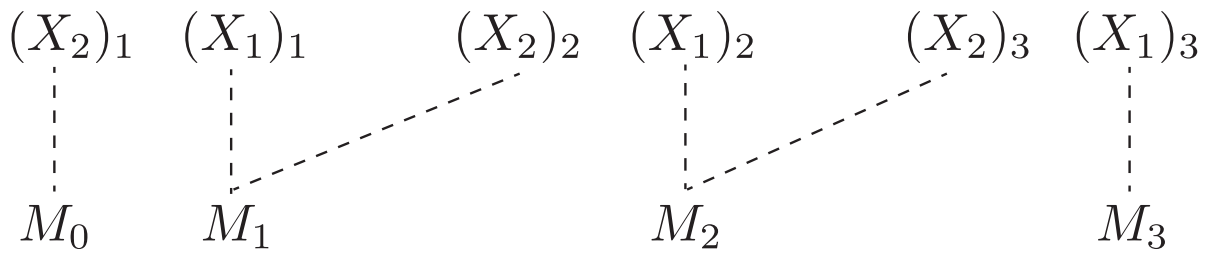
  - Index inheritance: For $v \in V$, $\mathrm{ind}(v') \subseteq \mathrm{ind}(v)$ for all $v' \in \mathrm{pa}(v)$

Definition of multiplex Bayesian network

We call the tuple $\mathcal{B} = (G, X, M, \text{ind})$, where $M \equiv \bigtimes_{j \in J} M_j$, a **multiplex Bayesian network**.

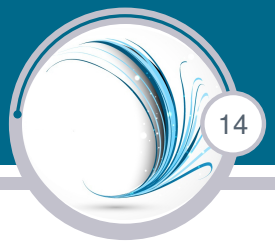
Visualization:

- Adjoin to G additional vertices M_j for each $j \in J$
- Connect with edge X_v to all M_j where $j \in \text{ind}(v)$



Quantum multiparty packing lemma

Multiplex Bayesian networks



Fixing multiplex Bayesian network (G, X, M, ind), generate a random codebook $\{x_v(m)\}_{v \in V, m \in M}$

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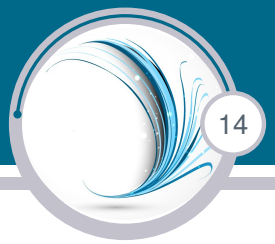

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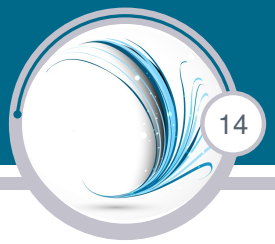

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Fixing multiplex Bayesian network (G, X, M, ind), generate a random codebook $\{x_v(m)\}_{v \in V, m \in M}$

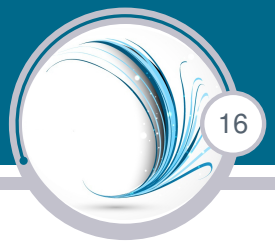
- x_v corresponds to vertices of G
- M is message set
- $x_v(m)$ only depends on m_j where $j \in \text{ind}(v)$

Algorithm for generating random codebook:

for $v \in V$ **do**
  for $m_v \in M_{\mathrm{ind}(v)}$ **do**
    generate $x_v(m_v)$ according to $p_{X_v|X_{\mathrm{pa}(v)}}\left(\cdot|x_{\mathrm{pa}(v)}(m_{\mathrm{pa}(v)})\right)$
    for $m_{\bar{v}} \in M_{\overline{\mathrm{ind}(v)}}$ **do**
      $x_v(m_v, m_{\bar{v}}) = x_v(m_v)$
    end for
  end for
end for

Quantum Multiparty Packing Lemma

Simultaneous Decoding



- How to find a decoder?

Quantum Multiparty Packing Lemma

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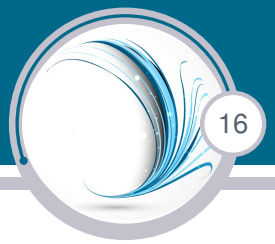

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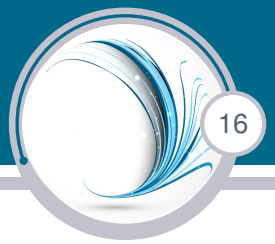

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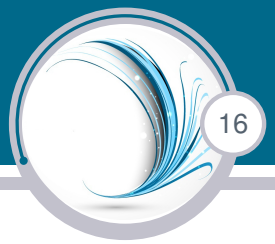

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- Simultaneous decoder!
  - Needs quantum joint typicality

Quantum Multiparty Packing Lemma

Sen's quantum joint typicality

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- Recent breakthrough: Pranab Sen, arXiv 1806.0727 "A one-shot quantum joint typicality lemma"
  - [Insert complicated equations here.]

Quantum Multiparty Packing Lemma

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Typicality $\Longleftrightarrow$ Packing lemma $\Longleftrightarrow$ Capacity theorem

Quantum multiparty packing lemma
Packing lemma statement

- Let C be codebook generated by multiplex Bayesian network

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Packing lemma statement

- Let C be codebook generated by multiplex Bayesian network
- Let $\{\rho_B^{(x)}\}_{x\in\mathcal{X}}$ be family of quantum states
  - Defines the black box
- Let $D \subseteq J$ be subset of messages to be decoded, $\overline{D}$ be guess for other messages

Asymptotic quantum multiparty packing lemma (simplified)

Let $\mathcal{B} = (G, X, M, \text{ind})$, $C^n = \{x^n(m)\}_{m \in M}$, $\{\rho_B^{(x)}\}_{x \in \mathcal{X}}$, $D \subseteq J$, and $\varepsilon \in (0, 1)$.

Then there exists a POVM $\{Q_B^{(m_D|m_{\overline{D}})}\}_{m_D \in M_D}$ for each $m_{\overline{D}} \in M_{\overline{D}}$ that, assuming the guess was right, can decode $(m_D, m_{\overline{D}}) \in M$ encoded into $\bigotimes_{i=1}^n \rho_{B_i}^{(x_i(m_D, m_{\overline{D}}))}$ with vanishing probability of error as $n \to \infty$ if

$$\forall\, \emptyset \neq T \subseteq D, \quad \sum_{t \in T} R_t < I(X_{S_T}; B|X_{\overline{S_T}})_\rho,$$

where $R_t \equiv \log |M_t|$, $S_T \equiv \{v \in V \mid \text{ind}(v) \cap T \neq \emptyset\} \subseteq V$,

$$\rho_{XB} \equiv \sum_{x \in \mathcal{X}} p_X(x) |x\rangle \langle x|_X \otimes \rho_B^{(x)}.$$

Conclusion
Summary

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  - Multiplex Bayesian networks
  - Cornerstone of classical network information theory is packing lemma
  - Allows hitherto impossible direct quantum generalization
- Demonstrated direct quantum generalization for classical-quantum relay channel
  - Multihop
  - Cohere multihop
  - Decode forward
  - Partial decode forward

Conclusion
Open questions

Open questions and future work:

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- Other notions of joint typicality?
- Rate region simplification?
- More general packing lemma?

Thank you!

Lemma (Classical packing lemma)

Let (U, X, Y) be a triple of random variables with joint distribution p_{UXY}. For each n, let $(\widetilde{U}^n, \widetilde{Y}^n)$ be a pair of arbitrarily distributed random sequences and $\{\widetilde{X}^n(m)\}$ a family of at most 2^{nR} random sequences such that each $\widetilde{X}^n(m)$ is conditionally independent of $\widetilde{Y}^n$ given $\widetilde{U}^n$. Further assume that each $\widetilde{X}^n(m)$ is distributed as $\otimes_{i=1}^n p_{X|U=\widetilde{U}_i}$ given $\widetilde{U}^n$. Then, there exists $\delta(\varepsilon)$ that tends to zero as $\varepsilon \to 0$ such that

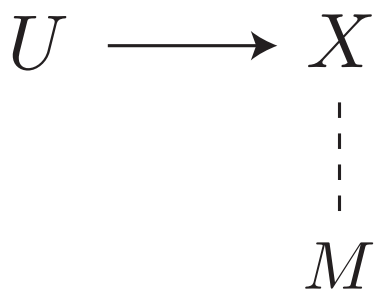
$$\lim_{n\to\infty} \Pr((\widetilde{U}^n, \widetilde{X}^n(m), \widetilde{Y}^n) \in \mathcal{T}_\varepsilon^{(n)} \text{ for some } m) = 0$$

if $R < I(X; Y|U) - \delta(\varepsilon)$, where $\mathcal{T}_\varepsilon^{(n)}$ is the set of ε-typical strings of length n with respect to p_{UXY}.

Usual channel coding follows from $U = \emptyset$ and $\widetilde{Y}^n \sim p_Y^{\otimes n}$.

Backup
Reduction to classical packing lemma

i.i.d. version of classical packing lemma is a corollary of our quantum multiparty packing lemma.



- Run algorithm n times, codebook generated is $\widetilde{U}^n, \widetilde{X}^n(m)$
- Set of quantum states
  $\left\{\rho_{\widetilde{Y}}^{(u,x)} \equiv \sum_{\widetilde{y}\in\mathcal{Y}} p_{Y|UX}(\widetilde{y}|u,x)\,|\widetilde{y}\rangle\,\langle\widetilde{y}|_{\widetilde{Y}}\right\}_{u\in\mathcal{U},x\in\mathcal{X}}$
- "Typicality test" POVM $\left\{Q_{\widetilde{Y}^n}^{(m)}\right\}_{m\in M}$

Backup

Sen's quantum joint typicality lemma (baby version)

Let $B_1 \dots B_k$ be a k-partite quantum system with each B_i isomorphic to a Hilbert space $\mathcal{H}$. Let $\rho_{B_1 \dots B_k}$ be a quantum state in $B_1 \dots B_k$. For a subset $S \subseteq [k]$, let B_S denote the systems $\{B_s : s \in S\}$. Let ρ_{B_S} denote the marginal state on B_S obtained by tracing out the systems in $\bar{S} := [k] \setminus S$ from $\rho_{B_1 \dots B_k}$. Let $0 < \epsilon < 1$. Let $\mathcal{K}$ be a Hilbert space of dimension $d_{\mathcal{K}}$ There exist a state $\tau_{\mathcal{K} \otimes [k]}$ independent of $\rho_{B_{[k]}}$, a state $(\rho')_{B'_{[k]}}$, and a POVM element $\Pi'_{B'_{[k]}}$ on $B'_1 \dots B'_k$ where $B'_i \cong B_i \otimes \mathcal{K}$, with the following properties:

1. $\|(\rho')_{B'_{[k]}} - \rho_{B_{[k]}} \otimes \tau_{\mathcal{K} \otimes [k]}\|_1 \leq f(k, \varepsilon)$;
2. $\mathrm{tr}[(\Pi')_{B'_{[k]}} (\rho')_{B'_{[k]}}] \geq 1 - g(k, \varepsilon)$;
3. For every set S, $\{\} \neq S \subset [k]$,

$$\mathrm{tr}[(\Pi')_{B'_{[k]}} ((\rho')_{B'_S} \otimes (\rho')_{B'_{\bar{S}}})] \leq 2^{-D_H^{\epsilon}(\rho_{B_{[k]}} \| \rho_{B_S} \otimes \rho_{B_{\bar{S}}}) \cdot + h(k, d_{\mathcal{H}}, d_{\mathcal{K}})}$$

From Pranab Sen, "A one-shot quantum joint typicality lemma"