Alpha-bits, Teleportation and Black Holes


Geoffrey Penington, Stanford University
Alpha-bits: Teleportation and Black Holes


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- Quantum error correction in AdS/CFT is only approximate and bulk operators are state-dependent.
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- Quantum error correction in AdS/CFT is only approximate and bulk operators are state-dependent.
- It solves the black hole information paradox?
Part I: Alpha-bits and Teleportation
Quantum Communication Resource
Inequalities
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Inequalities

1 qubit $\geq$ 1 ebit
Quantum Communication Resource Inequalities

\[ 1 \text{ cbit} \leq 1 \text{ qubit} \geq 1 \text{ ebit} \]
Quantum Communication Resource
Inequalities

\[ 1 \text{ cbit} \leq 1 \text{ qubit} \geq 1 \text{ ebit} \]

\[ 1 \text{ ebit} + 2 \text{ cbits} \geq 1 \text{ qubit} \]
Quantum Communication Resource

Inequalities

\[ 1 \text{ cbit} < 1 \text{ qubit} > 1 \text{ ebit} \]

\[ 1 \text{ ebit} + 2 \text{ cbits} > 1 \text{ qubit} \]
Quantum Communication Resource
Inequalities

1 cbit < 1 qubit > 1 ebit

zero-bits

1 ebit + 2 ebits > 1 qubit
Quantum Communication Resource Inequalities

1 cbit $< 1$ qubit $> 1$ ebit

different version of zero-bits

1 ebit + 2 ebits $> 1$ qubit

weakened version of qubits
Quantum Communication Resource Inequalities

\[ 1 \text{ cbit} < 1 \text{ qubit} > 1 \text{ ebit} \]

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weakened version of qubits
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\[ m \text{ qubits} \geq 2m \text{ zero-bits} \overset{(a)}{=} \]

\[ 1 \text{ cbit} > 1 \text{ zero-bit} ? \]

weakened version of qubits

asymptotic
Quantum Communication Resource Inequalities

\[
1 \text{ cbit} < 1 \text{ qubit} > 1 \text{ ebit}
\]

\[
1 \text{ ebit} + 2 \text{ zero-bits} \overset{(a)}{=} 1 \text{ qubit}
\]

\[
m \text{ qubits} \geq 2m \text{ zero-bits} \overset{(a)}{?}
\]

\[
1 \text{ cbit} > 1 \text{ zero-bit} ?
\]
What are zero-bits?
What are zero-bits?

\[ 1 \text{ zero-bit} = 1 \alpha\text{-bit with } \alpha = 0 \]
What are zero-bits?
What are zero-bits?

\[ |\psi\rangle \xrightarrow{U} |\psi^E\rangle \approx |\omega^E\rangle \]
What are zero-bits?

\[ |\psi\rangle \quad \begin{array}{ccc}
\text{U} \\
\hline \\
\text{B} \\
\hline \\
\text{E} \\
\hline \\
\text{dB} \gg d_E
\end{array} \approx \omega^E \]
What are zero-bits?
What are zero-bits?
What are zero-bits?

$d_B >> d_E$

\[ \forall |\psi_1\rangle, |\psi_2\rangle, \]
\[ \|\psi_1^B - \psi_2^B\|_1 \approx \|\psi_1 - \psi_2\|_1 \]

\[ \forall |\psi\rangle, \]
\[ \psi^E \approx \omega^E \]
What are zero-bits?

\[ d_B \gg d_E \]

∀ |ψ₁⟩, |ψ₂⟩,
\[ \|ψ_1^B - ψ_2^B\|_1 \approx \|ψ_1 - ψ_2\|_1 \]

∀ |ψ⟩,
\[ ψ^E \approx ω^E \]
What are zero-bits?

\[ d_B \gg d_E \]

\[ \forall |\psi_1\rangle, |\psi_2\rangle, \quad \|\psi_1^B - \psi_2^B\|_1 \approx \|\psi_1 - \psi_2\|_1 \]

\[ \forall |\psi\rangle, \quad \psi^E \approx \omega^E \]
What are zero-bits?

$B$ encodes the zero-bits of $|\psi\rangle$

$d_B >> d_E$

∀ $|\psi_1\rangle, |\psi_2\rangle$

$$\|\psi_1^B - \psi_2^B\|_1$$

≈ $$\|\psi_1 - \psi_2\|_1$$

∀ $|\psi\rangle$

$\psi^E \approx \omega^E$
What can you do with zero-bits?
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Not possible to recover the state $|\psi\rangle$ from B with no information about the state
- Error correction is not possible
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Able to error-correct any two-dimensional subspace $S$
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Able to error-correct any two-dimensional subspace $S$
Definition of zero-bits
Definition of qubits

“$n$ qubits”
Definition of qubits

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$\mathcal{N} : S(A) \rightarrow S(B)$

$d_A = 2^n$
Definition of qubits

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$d_A = 2^n$

What do we need to be true about the channel?
Definition of qubits

“$n$ qubits”

$\mathcal{N} : S(A) \rightarrow S(B)$

$d_A = 2^n$

What do we need to be true about the channel?

$\mathcal{N} = \text{Id}$ ?
Definition of qubits

“$n$ qubits”

$N : S(A) \rightarrow S(B)$

$d_A = 2^n$

$\exists \mathcal{D}$

$\mathcal{D} \circ N = \text{Id}$

What do we need to be true about the channel?

Bob can always error correct so long as error correction is possible
Definition of zero-bits

"n zero-bits"

\[ \mathcal{N} : S(A) \rightarrow S(B) \]

\[ d_A = 2^n \]

\[ \forall S \subseteq A \quad d_S = 2 \]

\[ \exists \mathcal{D}_S \quad \mathcal{D}_S \circ \mathcal{N}|_S = \text{Id}_S \]

OK now what about zero-bits?

Now Bob only has to be able to error correct any \textit{two-dimensional subspace}
Definition of zero-bits

"n qubits"

\[ \mathcal{N} : S(A) \rightarrow S(B) \]

\[ d_A = 2^n \]

\[ \forall S \subseteq A \quad d_S = 2 \]

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Definition of zero-bits

“$n$ zero-bits”

$\mathcal{N} : S(A) \to S(B)$

$d_A = 2^n$

$\forall S \subseteq A \quad d_S = 2$

$\exists \mathcal{D}_S$

$\| \mathcal{D}_S \circ \mathcal{N} \|_S - \text{Id}_S \|_S \leq \delta$

Need to make definition approximate if zero-bits are to be different from qubits
Definition of zero-bits

“n zero-bits”

\[ \mathcal{N} : S(A) \rightarrow S(B) \]

\[ d_A = 2^n \]

\[ \forall S \subseteq A \quad d_s = 2 \]

\[ \exists \mathcal{D}_s \]

\[ \| \mathcal{D}_s \circ \mathcal{N} |_S - \text{Id}_S \|_\diamond \leq \delta \]

[Hayden, Winter 2012]
Definition of zero-bits

"n zero-bits"

\[ \mathcal{N} : S(A) \rightarrow S(B) \]

\[ \mathcal{N}^c : S(A) \rightarrow S(E) \]

[Hayden, Winter 2012]

\[ d_A = 2^n \]

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Definition of zero-bits

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\[ \mathcal{N}^c : S(A) \rightarrow S(E) \]

\[ \forall |\psi\rangle \in A, \quad \| \mathcal{N}^c(\psi - \omega) \|_1 \leq \varepsilon \]

[Hayden, Winter 2012]
Definition of zero-bits

“\( n \) zero-bits”

\[ \mathcal{N} : S(A) \rightarrow S(B) \]

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\[ \exists \mathcal{D}_S \]

\[ \| \mathcal{D}_S \circ \mathcal{N} \|_S - \text{Id}_S \|_\diamond \leq \delta \]

\[ \mathcal{N}^c : S(A) \rightarrow S(E) \]

\[ \forall |\psi\rangle \in A, \quad \| \mathcal{N}^c (|\psi\rangle - \omega) \|_1 \leq \varepsilon \]

\[ \frac{1}{16} \delta^2 \leq \varepsilon \leq 8 \sqrt{\delta} \]
Definition of alpha-bits

"n α-bits"

\[ \mathcal{N} : S(A) \rightarrow S(B) \]

\[ d_A = 2^n \]

\[ \forall S \subseteq A \quad d_S \leq 2^{\alpha n} + 1 \]

\[ \exists \mathcal{D}_S \]

\[ \| \mathcal{D}_S \circ \mathcal{N} \|_S - \|\text{Id}_S\|_\diamond \leq \delta \]
Definition of alpha-bits

"n \alpha\text{-bits}"

\[\mathcal{N} : S(A) \rightarrow S(B)\]

\[d_A = 2^n\]

\[\forall S \subseteq A \quad d_S \leq 2^{\alpha n} + 1\]

\[\exists \mathcal{D}_S \quad \alpha = 1 \Rightarrow \text{qubits}\]

\[\|\mathcal{D}_S \circ \mathcal{N}\|_S - \text{Id}_S\|_\diamond \leq \delta\]
Definition of alpha-bits

“\( n \ \alpha\)-bits”

\[ \mathcal{N} : S(A) \to S(B) \]

\[ d_A = 2^n \]

\[ \forall S \subseteq A \quad d_S \leq 2^{\alpha n} + 1 \]

\[ \exists \mathcal{D}_S \quad \alpha = 1 \Rightarrow \text{qubits} \]

\[ \| \mathcal{D}_S \circ \mathcal{N} |_S - \text{Id}_S \|_\diamond \leq \delta \]

\[ \mathcal{N}^c : S(A) \to S(E) \]

\[ \forall \left| \psi \right\rangle \in \mathcal{A} \mathcal{R}, \quad d_R = 2^{\alpha n} \quad \| \mathcal{N}^c \otimes \text{Id}_R \left( \psi - \omega \otimes \psi^R \right) \|_1 \leq \varepsilon \]

\[ \frac{1}{16} \delta^2 \leq \varepsilon \leq 8\sqrt{\delta} \]
 Transmitting alpha-bits

\[ A \xrightarrow{n} U \xrightarrow{\beta n} B \]

\[ U \xrightarrow{(1 - \beta) n} E \]

\[ R \xrightarrow{\alpha n} \]
Transmitting alpha-bits

Necessary condition to send alpha-bits. Also sufficient (with some subtleties about needing to use shared randomness and block coding).
Transmitting alpha-bits

\[ \beta > \frac{1 + \alpha}{2} \]

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Transmitting alpha-bits

\[ \frac{1 + \alpha}{2} \ n \ \text{qubits} \]

\[ A \]
\[ U \]
\[ \{ \}
\[ B \]
\[ \{ \beta n \}
\[ \{ (1 - \beta) n \} \]
\[ E \]
\[ R \]
\[ \{ \alpha n \} \]

\[ \beta > \frac{1 + \alpha}{2} \]
Transmitting alpha-bits

\[ \frac{1 + \alpha}{2} n \text{ qubits} \geq n \text{ } \alpha\text{-bits} \]
Transmitting alpha-bits

\[ \frac{1 + \alpha}{2} n \text{ qubits} \geq n \alpha \text{-bits} \]

Diagram:

- \( A \) \( n \) \( U \)
- \( R \) \( \alpha n \)
- \( B \) \( \beta n \)
- \( A' \) \( (1 - \beta) n \)

\[ \beta > \frac{1 + \alpha}{2} \]
Transmitting alpha-bits

\[ \frac{1 + \alpha}{2} n \text{ qubits} \geq n \alpha\text{-bits} + \frac{1 - \alpha}{2} n \text{ ebits} \]

\[ A \]

\[ B \]

\[ \beta > \frac{1 + \alpha}{2} \]
Transmitting alpha-bits

\[(1 + \alpha) \text{ qubits} \overset{(a)}{\geq} 2 \alpha\text{-bits} + (1 - \alpha) \text{ ebits}\]

\[
\begin{array}{c}
A \\
\downarrow^n \\
\uparrow^n \\
\left\{ \begin{array}{c}
U \\
\downarrow^n \\
\uparrow^n \\
\left\{ \begin{array}{c}
\beta n \\
B \\
\end{array} \right. \\
\downarrow^n \\
\uparrow^n \\
\left\{ \begin{array}{c}
(1 - \beta) n \\
A' \\
\end{array} \right. \\
\downarrow^n \\
\uparrow^n \\
\left\{ \begin{array}{c}
\alpha n \\
R \\
\end{array} \right. \\
\end{array} \right. \\
\end{array}
\]

\[\beta > \frac{1 + \alpha}{2}\]
Alpha-bit resource equalities

\[(1 + \alpha) \text{ qubits} \overset{(a)}{=} 2 \alpha\text{-bits} + (1 - \alpha) \text{ ebits}\]
Alpha-bit resource equalities

\[(1 + \alpha)\ \text{qubits} \overset{(a)}{=} 2\ \alpha\text{-bits} + (1 - \alpha)\ \text{ebits}\]

To show \(\text{RHS} \geq \text{LHS}\):

1. \(1\ \alpha\text{-bit} + 1\ \text{ebit} \overset{(a)}{\geq} (1 + \alpha)\ \text{cobits}\)
Alpha-bit resource equalities

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Alpha-bit resource equalities

\[(1 + \alpha) \text{ qubits} \overset{(a)}{=} 2 \alpha \text{-bits} + (1 - \alpha) \text{ ebits}\]

To show RHS $\geq$ LHS:

1. $1 \alpha$-bit + 1 ebit $\overset{(a)}{\geq} (1 + \alpha)$ cobits

2. 1 qubit + 1 ebit $= 2$ cobits \(\checkmark\)
Alpha-bit resource equalities

\[(1 + \alpha) \text{ qubits} \overset{(a)}{=} 2 \alpha\text{-bits} + (1 - \alpha) \text{ ebits}\]

To show RHS \(\geq\) LHS:

1. \(1 \alpha\text{-bit} + 1 \text{ ebit} \overset{(a)}{\geq} (1 + \alpha) \text{ cobits}\)

2. \(1 \text{ qubit} + 1 \text{ ebit} = 2 \text{ cobits}\)
What is a cobit?
What is a cobit?

Classical bit:

\[ |0\rangle_A \rightarrow |0\rangle_B |0\rangle_E \quad |1\rangle_A \rightarrow |1\rangle_B |1\rangle_E \]
What is a cobit?

Classical bit:

\[ |0\rangle_A \rightarrow |0\rangle_B |0\rangle_E \]
\[ |1\rangle_A \rightarrow |1\rangle_B |1\rangle_E \]

Coherent classical bit (cobit):

\[ |0\rangle_A \rightarrow |0\rangle_B |0\rangle_A \]
\[ |1\rangle_A \rightarrow |1\rangle_B |1\rangle_A \]

Alice keeps purification
(Coherent) super-dense coding
(Coherent) super-dense coding

1. Alice and Bob share $n$ ebits

$$
\sum_{k=0}^{2^n-1} |k\rangle_A |k\rangle_B
$$
(Coherent) super-dense coding

1. Alice and Bob share $n$ ebits
2. Alice applies an operation to her qubits

$$
\sum_{k=0}^{2^n-1} e^{\frac{2\pi k r i}{2^n}} |k \oplus s\rangle_A |k\rangle_B
$$

$$
0 \leq r < 2^n \\
0 \leq s < 2^n
$$
(Coherent) super-dense coding

1. Alice and Bob share $n$ ebits
2. Alice applies an operation to her qubits
3. Alice sends her qubits to Bob

\[
\sum_{k=0}^{2^n-1} e^{\frac{2\pi kr i}{2^n}} |k \oplus s\rangle_A |k\rangle_B
\]

$0 \leq r < 2^n$

$0 \leq s < 2^n$
(Coherent) alpha-bit super-dense coding

1. Alice and Bob share $n$ ebits
2. Alice applies an operation to her qubits

$$\sum_{k=0}^{2^n-1} e^{\frac{2\pi k r i}{2^n}} |k \oplus s\rangle_A |k\rangle_B$$

$$0 \leq r < 2^n$$
$$0 \leq s < 2^{\alpha n}$$
(Coherent) alpha-bit super-dense coding

1. Alice and Bob share $n$ ebits
2. Alice applies an operation to her qubits
3. Alice sends her qubits to Bob as $\alpha$-bits

$$
\sum_{k=0}^{2^n-1} e^{\frac{2\pi k r i}{2^n}} |k \oplus s\rangle_A |k\rangle_B
$$

$0 \leq r < 2^n$
$0 \leq s < 2^{\alpha n}$
(Coherent) alpha-bit super-dense coding

1. Alice and Bob share $n$ ebits
2. Alice applies an operation to her qubits
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$$\sum_{k=0}^{2^n-1} e^{\frac{2\pi k r i}{2^n}} |k \oplus s\rangle_A |k\rangle_B$$

$$0 \leq r < 2^n$$
$$0 \leq s < 2^{\alpha n}$$

Done
Zero-bits and ebits as fundamental resources

All noiseless quantum resources (qubits, \( \alpha \)-bits, cobits . . . ) can be rewritten in terms of zero-bits and ebits

\[
\text{e.g. } 1 \ \alpha\text{-bit} \overset{(a)}{=} (1 + \alpha) \text{ zero-bits} + \alpha \text{ ebits}
\]
Zero-bits and ebits as fundamental resources

All noiseless quantum resources (qubits, \( \alpha \)-bits, cobits \ldots) can be rewritten in terms of zero-bits and ebits

\[ 1 \alpha\text{-bit} \overset{(a)}{=} (1 + \alpha) \text{ zero-bits} + \alpha \text{ ebits} \]

When rewritten in this basis, the quantum resource ordering becomes the product ordering:

\[ (a, b) \geq (a', b') \iff (a \geq a') \land (b \geq b') \]
Alpha-bit Capacities

The $\alpha$-bit capacity of a channel $\mathcal{N} : S(A') \to S(B)$ is given by

$$Q_\alpha(\mathcal{N}) = \sup_k \frac{1}{k} \sup_{|\psi\rangle \in A'^k A^k} \min \left( \frac{1}{1 + \alpha} I(A : B)_\rho, \frac{1}{\alpha} I(A\rangle B)_\rho \right)$$

where $\rho = (\mathcal{N}^\otimes k \otimes \text{Id})\psi$
Alpha-bit Capacities

The $\alpha$-bit capacity of a channel $\mathcal{N} : S(A') \rightarrow S(B)$ is given by

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where $\rho = (\mathcal{N}^\otimes k \otimes \text{Id}) |\psi\rangle$.

1 $\alpha$-bit $\overset{(a)}{=} (1 + \alpha)$ zero-bits + $\alpha$ ebits

- zero-bit limited
- ebit limited
Amortised and entanglement-assisted capacities

With entanglement-assistance or an amortised quantum side channel, the capacity is given by

$$\frac{1}{1 + \alpha} \sup_{|\psi\rangle \in A' A} I(A : B)_\rho$$

Single letter!

Unconstrained by ebits and so only zero-bits matter. This explains why all entanglement-assisted capacities are proportional to mutual information.
Amortised and entanglement-assisted capacities

With entanglement-assistance or an amortised quantum side channel, the capacity is given by

\[
\frac{1}{1 + \alpha} \sup_{|\psi\rangle \in A' \mathcal{A}} I(A : B)_\rho
\]

Unconstrained by ebits and so only zero-bits matter. This explains why all entanglement-assisted capacities are proportional to mutual information.

If \(\alpha \to 1\) amortised \(\alpha\)-bit capacity \(\to\) entanglement-assisted quantum capacity

BUT amortised quantum capacity = quantum capacity
Amortised and entanglement-assisted capacities

With entanglement-assistance or an amortised quantum side channel, the capacity is given by

\[
\frac{1}{1 + \alpha} \sup_{|\psi\rangle \in A'\Lambda} I(A : B)_{\rho}
\]

Unconstrained by ebits and so only zero-bits matter. This explains why all entanglement-assisted capacities are proportional to mutual information.

If \( \alpha \to 1 \) amortised \( \alpha \)-bit capacity \( \Rightarrow \) *entanglement – assisted* quantum capacity

**BUT** amortised quantum capacity = quantum capacity

**Answer:** As \( \alpha \to 1 \), the size of the side channel diverges
Further Applications

\[ \langle \mathcal{N}_{A' \to B} \rangle \stackrel{(a)}{\geq} I(A \parallel B) \text{ qubits} \]
Further Applications

Non-additivity of quantum capacity?

$$\langle N_{A' \to B} \rangle^{(a)} \geq I(A\rightarrow B) \text{ qubits} + I(A; E) \text{ zero-bits}$$
Further Applications

Non-additivity of quantum capacity?

\[ \langle \mathcal{N}_{A' \rightarrow B} \rangle \overset{(a)}{\geq} I(A \mid B) \text{ qubits } + I(A; E) \text{ zero-bits} \]

Zero-bits can substitute for classical bits in:

- entanglement distillation, state merging
- remote state preparation and channel simulation

Optimality follows from optimality of zero-bit teleportation
Part II: Alpha-bits and Black Holes
AdS/CFT

Duality between an ordinary quantum field theory, specifically a CFT, known as the ‘boundary’ theory, and quantum gravity in asymptotically anti-de Sitter space in one higher dimension, the ‘bulk’.
AdS/CFT

Duality between an ordinary quantum field theory, specifically a CFT, known as the ‘boundary’ theory, and quantum gravity in asymptotically anti-de Sitter space in one higher dimension, the ‘bulk’.

What does this have to do with quantum information? Also what does it have to do with our universe which is not anti-de Sitter space?
The Ryu-Takayanagi formula
The Ryu-Takayanagi formula

\[ S(\rho_A) = \min_\Sigma \left[ \frac{\text{Area}}{4G_N} + S_{\text{bulk}}(\rho_{\alpha}) \right] \]
The Ryu-Takayanagi formula

\[ S(\rho_A) = \min_{\Sigma} \left[ \frac{\text{Area}}{4G_N} + S_{\text{bulk}}(\rho_a) \right] \]

“Information = Geometry”
Bulk operators in the central region can be represented by a boundary operator acting only on any two of the three boundary regions A, B and C.
Error correction and AdS/CFT

Bulk operators in the central region can be represented by a boundary operator acting only on any two of the three boundary regions A, B and C

(Operator algebra) quantum error correction
Error correction and AdS/CFT

Bulk operators in the central region can be represented by a boundary operator acting only on any two of the three boundary regions A, B and C.

(Operator algebra) quantum error correction

Bulk states with some particular geometry = code subspace of larger boundary Hilbert space.
Entanglement Wedge Reconstruction
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Conventional techniques only allow operators in top and bottom region to be reconstructed in region A.
Entanglement Wedge Reconstruction

Conventional techniques only allow operators in top and bottom region to be reconstructed in region A.

BUT Ryu-Takayanagi formula suggests region A ‘knows’ about the area of the solid line
Entanglement Wedge Reconstruction

Conventional techniques only allow operators in top and bottom region to be reconstructed in region A.

BUT Ryu-Takayanagi formula suggests region A ‘knows’ about the area of the solid line

Entanglement wedge reconstruction conjecture: Actually the entire region between A and the RT surface can be reconstructed
The DHW Proof of Entanglement Wedge Reconstruction

\[ \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}} \subseteq \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}} \]
The DHW Proof of Entanglement Wedge Reconstruction

RT formula implies:

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Exact zero-bits = exact qubits
Approximate zero-bits ≠ approximate qubits
The (slightly corrected) DHW Proof of Entanglement Wedge Reconstruction
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Theorem by Cedric Beny:

\[ \exists D \quad \| D \circ \text{Tr}_A - \text{Tr}_{\bar{a}} \|_\diamond \leq \delta \quad \iff \quad \forall |\psi\rangle \in \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}} \otimes \mathcal{H}_R \quad \| \text{Tr}_A (\omega_a \otimes \psi_{\bar{a}R}) - \psi_{\bar{A}R} \|_1 \leq \varepsilon \]

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Bulk operators that we want to reconstruct must lie within the entanglement wedge of A, even for states that are entangled with a reference system
Black holes and Alpha-bit Codes
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REMINDER:
$$\min \sum \left[ \frac{\text{Area}}{4G_N} + S_{\text{bulk}}(\rho a) \right]$$

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For states entangled with a reference system, region $a'$ is not always contained in the entanglement wedge if the reference system dimension

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We can only reconstruct operators if we know that the state lies in a sufficiently small subspace: the reconstruction is ‘state-dependent’

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Tensor Network Toy Models:
The HaPPY Code with a black hole
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Perfect tensor = unitary map from any three legs to the other three legs.
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Black hole described by a random unitary with one large ‘bulk’ leg and many outward-flowing legs
Alpha-bits in the HaPPY Code
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Operators in the equivalent of region a’ can only be pushed to region A by pushing them through the black hole
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Only possible if the black hole bulk leg is sufficiently small
Consequences of Alpha-bit Codes in AdS/CFT
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Operator reconstruction is state-dependent. State-dependence is also believed (by some) to be necessary to describe operators behind a black hole horizon.
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Hawking radiation is thermally entangled with $\mathcal{H}_A$ and so the entanglement entropy increases (agrees with RT formula).
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Also implies reconstruction is only approximate.
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By carefully analysing the location in spacetime of the covariant Ryu-Takayanagi surface, we find that information falling into the black hole appears in the Hawking radiation after exactly the scrambling time (Hayden-Preskill).
Thank you